Loop-Transfer Recovery via Hardy-Space Optimization

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A Hardy-space optimization technique is adapted to solve the Linear-Quadratic-Gaussian with Loop-Transfer-Recovery problem. Repetitive computations for the sequence of filter or controller gains as some parameter approaches zero or infinity are avoided. The limit values of the Loop-Transfer-Recovery compensator are obtained by the Hardy-space optimization in one iteration without involving infinite filter or controller gains.

Introduction

THE Linear-Quadratic-Gaussian synthesis with Loop-Transfer-Recovery (LQG/LTR) is an elegant method for achieving desired loop shapes and maximum robustness properties in the design of feedback control systems.[1,2] It involves essentially a two-step approach. First, a Kalman filter [or alternatively, a full-state Linear Quadratic (LQ) feedback regulator] with desired loop-transfer properties is designed. Then, a sequence of LQ feedback regulators (or alternatively, Kalman filters) approaching an ideal limit is designed and the combined Linear-Quadratic-Gaussian (LQG) compensator is selected with trade-offs.

This design method is powerful in that for minimum phase systems, it offers essentially arbitrary freedom to shape the loop-transfer characteristics in the high gain region, and it yields stability margins approaching those of the Kalman filter (or full-state regulator) in the limit. The compensator obtained from this design also retains the high-frequency roll-off characteristics of an LQG controller. This design method is easy to carry out because it primarily involves repeated solutions of algebraic Riccati equations. However, if the performance index approaches zero or infinity in the limit, when computing the sequence of compensators as a weighting factor, numerical difficulty may occur before the limit solution becomes apparent. In the numerical experiments conducted, it was observed that allowing the state weighting to approach infinity more often resulted in computational difficulty than driving the control weighting to zero when designing regulators to recover filter characteristics. The resultant sequence of compensators always involves some arbitrarily large values either in the regulator or Kalman filter gain matrix.

In a recent paper,[2] it was shown that the LQG integral performance index is equivalent to a two-norm over the Hardy space of stable rational transfer functions (H_2-space). Therefore, the LQG method (and, in particular, LQG/LTR procedures) may be regarded as one way of solving a particular H_2-optimization problem.

The solutions of H_2 and H_∞ (Hardy space with infinity norm) optimization were treated extensively in a recent dissertation.[3] The techniques presented enabled one to solve H_2-optimization problems directly in H_2-space without resorting to time-domain optimization.

In this paper, the H_2-optimization technique developed in Ref. 3 is adapted to solve the LQG/LTR problem in the limit. The objectives are twofold: 1) to obtain the LQG/LTR compensator without involving infinite gains in the filter or controller and 2) to avoid repetitive computations for the sequence of filter or controller gains in the recovery procedure and the numerical errors that one may associate with such a procedure.

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LQG/LTR Method vs H_2-Optimization

In LQG control (see Fig. 1), one selects the Kalman filter gain matrix K_f and LQ controller gain matrix K_c to minimize

$$J_{\mathrm{LQG}} = E\left\{ \lim_{T\to\infty} \frac{1}{T}\int_0^T \left[x^T(t)\, H_0^T H_0 x(t) + \rho^2 u^T(t)\, u(t)\right] \mathrm{d}t \right\} \tag{1}$$

where superscript T denotes transpose, H_0 is a constant column matrix, ρ is a constant scalar, u and x are control and state vectors, respectively. The gain matrices K_f and K_c are obtained from the solutions of two Riccati equations in which the strengths of the Gaussian white noises $\xi(t)$ and $\eta(t)$ enter as parameters. The LQG control does not have guaranteed stability margins and does not enable the engineer to choose the loop shape.

In the LQG/LTR problem, $\xi(t)$ and $\eta(t)$ no longer represent real noises but are treated as design parameters. They are assumed to have unity power spectral density so that the freedom in the parameter selection is given to the matrix L and the coefficient μ (see Fig. 1). That is, one selects L and μ such that

$$\frac{1}{\mu} C\Phi(s)\, L = W(s) \tag{2}$$

to achieve the desired loop shape $[W(s)]$. (In this paper, we consider the problem of breaking the loop at the output only. Breaking the loop at the input is a dual aspect of this problem[2] and can be similarly derived.) Also, H_0 is selected as in Eq. (3) and ρ is made to approach zero

$$H_0 = C; \qquad \rho \to 0 \tag{3}$$

so that in the limit, $G(s)K(s)$ approaches the Kalman filter loop-transfer matrix. The singular values of $G(s)K(s)$ approximate those of the desired loop shape $W(s)$ in the operating frequency range (due to the selection of L).

The right-hand side of Eq. (1) may be expressed in the frequency domain through Parseval's theorem. It has been shown[2] that under the conditions given by Eqs. (2) and (3), the performance index (1) may be represented by

$$J_{\mathrm{LQG}} = \frac{1}{\pi}\int_0^\infty Tr\left[M^H(j\omega)\, M(j\omega)\right] \mathrm{d}\omega = \| M(j\omega)\|_2^2 \tag{4}$$

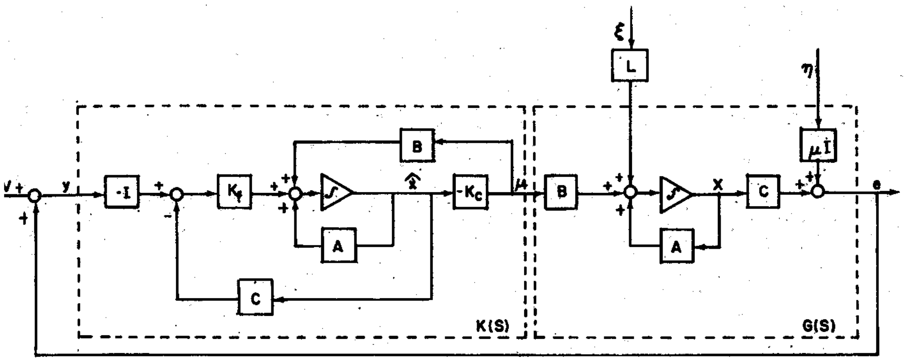

Fig. 1 Block diagram of LQG system.

where

$$M(s)=\mu\begin{bmatrix}[I+G(s)K(s)]^{-1}W(s) & -G(s)K(s)[I+G(s)K(s)]^{-1}\\ 0 & 0\end{bmatrix} \tag{5}$$

Thus, the LQG/LTR problem can also be solved by minimizing the L_2 norm of $M(s)$ in the H_2-space.

Note that if the optimization of Eq. (1) is carried out in the time domain subject to Eqs. (2) and (3) (the regular LQG/LTR procedure), then at least one component in K_c approaches infinity as ρ approaches zero in the limit. Replacing Eq. (3) with $H_0=qC$, $\rho=1$, $q\to\infty$ should yield the same result theoretically. However, some experiments show that allowing $q\to\infty$ tends to result in greater numerical instability in the computational process. In contrast, the H_2-optimization yields an equivalent controller that has a slightly different configuration than $K(s)$ (Fig. 1) and involves no infinite gain components, as will be shown in the subsequent development.

Formulation of the H_2-Optimization Problem through Linear Fractional Transformation

The H_2-optimization problem may be solved by either the frequency-domain projection method of Chang and Pearson[4] or the state-space method of Doyle.[3] While Chang and Pearson's method applies to plants having either equal number or more inputs than outputs, Doyle's method applies to plants having equal number or more outputs than inputs. In this paper, the application of Doyle's H_2-optimization method (which requires system representation in the form of linear fractional transformation) to the LQG/LTR problem is explored.

The system of Fig. 2 is obtained from setting $\eta(t)=0$, $C\Phi(s)L=W(s)$ and rearranging Fig. 1 into the "linear fractional transformation" format. Let

$$\tilde{v}=\begin{bmatrix}v\\ \xi\end{bmatrix} \tag{6}$$

The transfer function relationship between $\tilde{v}(s)$ and $e(s)$ may be derived as

$$e(s)=T(s)\tilde{v}(s) \tag{7}$$

where

$$T(s)=\big[G(s)K(s)[I+G(s)K(s)]^{-1} \quad [I+G(s)K(s)]^{-1}W(s)\big] \tag{8}$$

The comparison of Eq. (8) with Eq. (5) shows that

$$\mu\|T(s)\|_2=\|M(s)\|_2 \tag{9}$$

Thus, the LQG/LTR design of Fig. 1 may be achieved by H_2-optimization of the system of Fig. 2, if the H_2-norm of

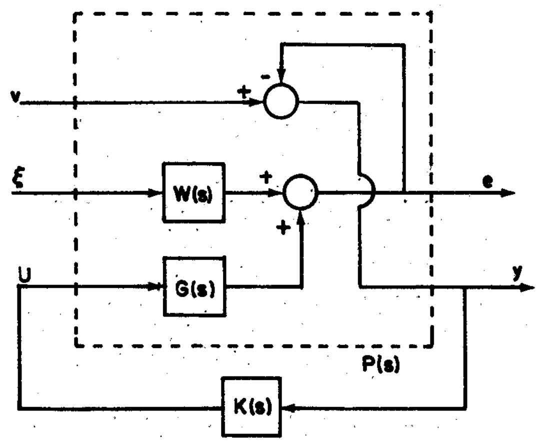

Fig. 2 Feedback system in linear fractional transformation form.

$T(s)$ is chosen to be minimized. The H_2-norm of a transfer function is an upper bound of its H_∞-norm, which is the maximum gain of the system when the signals are measured by H_2-norm.

The linear fractional transformation of Fig. 2 may be represented by

$$\begin{bmatrix}e(s)\\ y(s)\end{bmatrix}=P(s)\begin{bmatrix}\tilde{v}(s)\\ u(s)\end{bmatrix}=\begin{bmatrix}p_{11}(s) & p_{12}(s)\\ p_{21}(s) & p_{22}(s)\end{bmatrix}\begin{bmatrix}\tilde{v}(s)\\ u(s)\end{bmatrix} \tag{10}$$

where

$$p_{11}(s)=[0 \quad W(s)],\qquad p_{12}(s)=G(s),$$
$$p_{21}(s)=[I \quad -W(s)],\qquad p_{22}(s)=-G(s) \tag{11}$$

It has been shown by Doyle[3] that every compensator $K(s)$ that yields an internally stable linear fractional transformation (Fig. 2) can be represented by yet another linear fractional transformation shown in Fig. 3, where J can be computed from the parameters in any minimal realization of $P(s)$. That is,

$$P(s)=\begin{bmatrix}p_{11}(s) & p_{12}(s)\\ p_{21}(s) & p_{22}(s)\end{bmatrix}=\left[\begin{array}{c|cc}A_p & B_1 & B_2\\ \hline C_1 & D_{11} & D_{12}\\ C_2 & D_{21} & D_{22}\end{array}\right] \tag{12}$$

$$J(s)=\begin{bmatrix}J_{11}(s) & J_{12}(s)\\ J_{21}(s) & J_{22}(s)\end{bmatrix}$$
$$\times\left[\begin{array}{c|cc}A_p+B_2F+HC_2+HD_{22}F & -H & B_2+HD_{22}\\ \hline F & 0 & I\\ -(C_2+D_{22}F) & I & -D_{22}\end{array}\right] \tag{13}$$

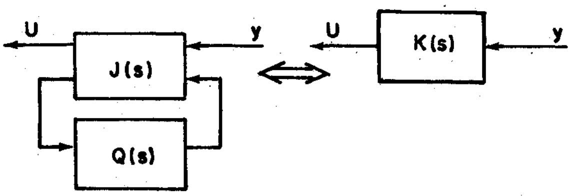

Fig. 3 Compensator as a linear fractional transformation.

The right-hand sides of Eqs. (12) and (13) are minimal realizations of $P(s)$ and $J(s)$, respectively. F and H are any real matrices (of compatible sizes) that stabilize $A_p + B_2 F$ and $A_p + HC_2$, respectively. They may be obtained by solving a pair of matrix Riccati equations in which some coefficients may be arbitrarily set.[5]

The block $Q(s)$ in Fig. 3 is any proper rational matrix of the same size as the transpose of $p_{22}(s)$ and is analytic and bounded in the right half-plane (asymptotically stable). The only other condition that $Q(s)$ must satisfy is that

$$\det[I + D_{22}Q(\infty)] \neq 0 \tag{14}$$

The H_2-Optimal Controller

It has been shown by Doyle[3] that if $p_{12}(s)$ and $p_{21}(s)$ have no transmission zeros on the $j\omega$ axis (including infinity) and if

$$D_{12}^T D_{12} = I \tag{15}$$

$$D_{21} D_{21}^T = I \tag{16}$$

then the controller $Q(s)$ that minimizes the H_2-norm of $T(s)$ is found to be

$$Q_{\text{opt}}(s) = -D_{12}^T D_{11} D_{21}^T \tag{17}$$

provided that F and H are selected as follows:

$$F = -\left(D_{12}^T C_1 + B_2^T X\right) \tag{18}$$

$$H = -\left(B_1 D_{21}^T + Y C_2^T\right) \tag{19}$$

where X and Y are the solutions of the following Riccati equations:

$$\left(A_p - B_2 D_{12}^T C_1\right)^T X + X\left(A_p - B_2 D_{12}^T C_1\right) - X B_2 B_2^T X + C_1^T D_\perp D_\perp^T C_1 = 0 \tag{20}$$

$$\left(A_p - B_1 D_{21}^T C_2\right) Y + Y\left(A_p - B_1 D_{21}^T C_2\right)^T - Y C_2^T C_2 Y + B_1 \tilde{D}_\perp^T \tilde{D}_\perp B_1^T = 0 \tag{21}$$

where $D_\perp$ is an orthogonal complement of D_{12} $[D_\perp \triangleq (D_{12})_\perp]$, and $\tilde{D}\perp$ is an orthogonal complement of D_{21} $[\tilde{D}_\perp \triangleq (D_{21})_\perp]$ such that

$$[D_{12} \quad D\perp]^T [D_{21} \quad D_\perp] = I \tag{22}$$

$$\begin{bmatrix} D_{21} \\ \tilde{D}_\perp \end{bmatrix} \begin{bmatrix} D_{21} \\ \tilde{D}_\perp \end{bmatrix}^T = I \tag{23}$$

A necessary condition for D_{12} to satisfy (15) is that D_{12} [and hence, $p_{12}(s)$] must have at least as many rows as columns. Therefore, the plant $G(s)$ must have at least as many outputs as inputs [see Eq. (11)].

It should be noted that for the case in which $p_{12}(s)$ has more columns than rows and $p_{21}(s)$ has more rows than columns, Chang and Pearson[4] have shown through the use of an example that a frequency-domain inner-outer factorization and H_2-projection method may be used to find the product $B_0(s)Q_{\text{opt}}(s)A_0(s)$, where $B_0(s)$ has a stable right inverse and $A_0(s)$ has a stable left inverse. In some special cases, $Q_{\text{opt}}(s)$ may be computed.

In this paper, $G(s)$ is assumed to have at least as many outputs as inputs so that Doyle's method is applicable. Since $p_{12}(s) = G(s)$, Eqs. (15) and (16) cannot be satisfied for practical plants that are usually strictly proper. Therefore, the H_2-optimization technique must be adapted to this case.

Manipulate Fig. 2 into the equivalent form shown in Fig. 4, where a nonsingular $G_p(s)$ is chosen so that $G(s)G_p(s)$ has a full rank D matrix and no transmission zeros on the imaginary axis. $G_p(s)$ does not introduce any right-half plane zeros to affect the projection results in the H_2-space and the R matrix is chosen so that Eq. (15) may be satisfied. One convenient form of $G_p(s)$ is a diagonal polynomial matrix.

Therefore, one may define

$$D_{12}^0 \equiv \lim_{s \to \infty} G(s) G_p(s) \tag{24}$$

$$R = \left[\left(D_{12}^0\right)^T \left(D_{12}^0\right)\right]^{1/2} = R^T \tag{25}$$

$$D_{12} = D_{12}^0 R^{-1} \tag{26}$$

If a minimal realization of $P(s)$ is written for the system of Fig. 4, then conditions (15) and (16) are satisfied because $W(s)$ is the desired loop shape that always vanishes at high frequency. The optimal controller $K_0(s)$ of Fig. 4 may be represented by Fig. 5 where $J(s)$ is given by Eq. (13) and $Q_{\text{opt}}(s)$ is given by Eq. (17). $Q_{\text{opt}}(s)$ vanishes for the LQG/LTR problem because

$$D_{11} = \lim_{s \to \infty} [0 \quad W(s)] = [0 \quad 0] \tag{27}$$

Thus, from Fig. 5 and by virtue of Eqs. (17) and (27), it follows that

$$K_0(s) = R G_p^{-1}(s) K(s) = J_{11}(s) \tag{28}$$

Equations (28) and (13) readily give the state-space configurations of the H_2-optimal compensator as shown in Fig. 6. As in the LTR procedure, the H_2-optimal compensator is always strictly proper. But mathematically, $K(s)$ may be proper or even improper depending on the loop shape $W(s)$ and the plant $G(s)$, because $G(s)K(s)$ must closely match $W(s)$. A proper or strictly proper controller $K(s)$ can always be guaranteed if the loop shape $W(s)$ is properly chosen to have a sufficient high-frequency roll-off characteristic. In any event, the parameters in H and F (see Fig. 6) are always finite. On the other hand, LQG/LTR compensators always involve arbitrarily large parameters in the gain matrices K_f or K_c.

It is worth noting that Fig. 6 also represents the H_∞-optimal control system, with the exception that $Q_{\text{opt}}(s)$ is no longer given by Eq. (17).

Example of Loop Recovery via H_2-Optimization

The mathematical development of the previous sections covers the general case of multiple-input multiple-output systems. A single-input single-output system is used in the numerical example of this section to compare the recovery of the desired loop-transfer function with H_2-optimization and the LQG/LTR procedure.

Referring to Fig. 1 for the system description, let

$$A = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mu = 1 \tag{29}$$

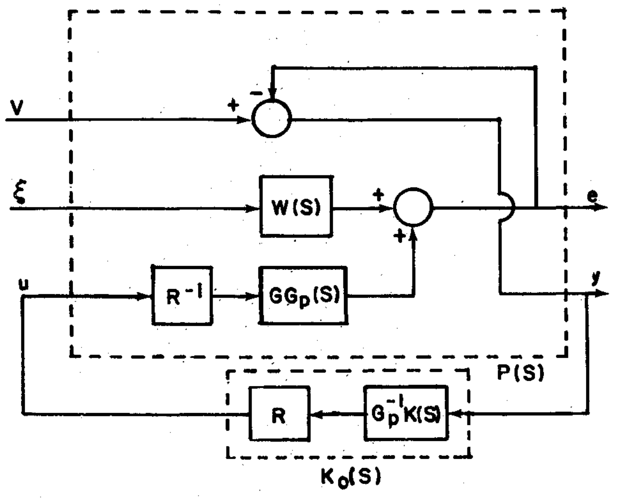

Fig. 4 Treatment of a strictly proper plant.

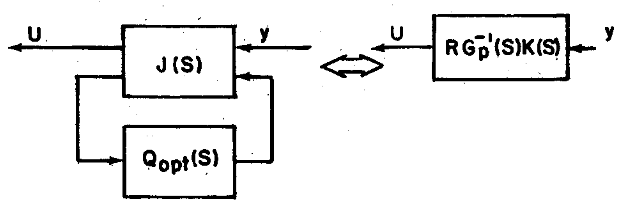

Fig. 5 Optimal compensator $K_0(s)$ of a strictly proper plant.

$$C = [2 \quad 1], \qquad Q_0 = 1, \qquad R_0 = 1 \tag{30}$$

where Q_0 and R_0 are the power spectral density of the uncorrelated Gaussian white noises $\xi(t)$ and $\eta(t)$, respectively. Thus,

$$G(s) = C(sI - A)^{-1}B = \frac{(s+2)}{(s+1)(s+3)} \tag{31}$$

and $W(s)$, the desired loop shape, is arbitrarily selected as

$$W(s) = \frac{10(s+5)}{(s+1)(s+2)} \tag{32}$$

The problem of how to select a loop shape is not an issue of this paper. Since $G(s)$ is strictly proper, select

$$G_p(s) = (s+4)I \tag{33}$$

The zero in $G_p(s)$ can be anywhere in the closed left half-plane without affecting the final result. This $G_p(s)$ yields $D_{12}(s) = 1$, $R = 1$, and $D_\perp = 0$. Equations (33) and (11) give $D_{21} = [1 \quad 0]$. Hence, $\tilde{D}_\perp = [0 \quad 1]$. Now, calculating the coefficient matrices in the right-hand side of Eq. (12) in accordance with Eq. (11) and using the results of Eqs. (18–21) yield

$$A_p = \begin{bmatrix} -3.4538 & 3.5364 \\ -0.3149 & -0.5462 \end{bmatrix} \qquad B_2 = \begin{bmatrix} -0.1182 \\ -0.0390 \end{bmatrix} \tag{34}$$

$$C_2 = F = [0 \quad 51.2738] \qquad D_{22} = [-1] \tag{35}$$

$$H = \begin{bmatrix} 0.6717 \\ -0.2047 \end{bmatrix} \qquad Q_{opt} = 0 \tag{36}$$

Finally, the controller transfer function is found to be

$$K(s) = G_p(s)R^{-1}K_0(s) = \frac{10.4976(s+4.4858)}{(s+2)} \tag{37}$$

To rework this problem via LQG/LTR procedure, choose

$$L = \begin{bmatrix} 30 \\ -50 \end{bmatrix} \qquad \mu = 1 \tag{38}$$

so that the right-hand side of Eq (2) equals that of Eq. (32). Following the standard LQG procedure yields a sequence of K_c and K_f as $\rho \to 0$. Specifically, for $\rho^2 = 10^{-12}$ (see Fig. 1),

$$K_c = [2 \quad 1] \times 10^6 \qquad K_f = \begin{bmatrix} 26.0948 \\ -41.6919 \end{bmatrix} \tag{39}$$

and

$$K(s) = \frac{10.498 \times 10^6(s+4.4853)}{(s+10^6)(s+2)} \tag{40}$$

Conclusions

A Hardy-space optimization technique has been adapted to solve the Linear-Quadratic-Gaussian with Loop-Transfer-Recovery problem. This method gives the limit solution of the Loop-Transfer-Recovery problem in one iteration without requiring any parameters to approach zero or infinity in the limit.

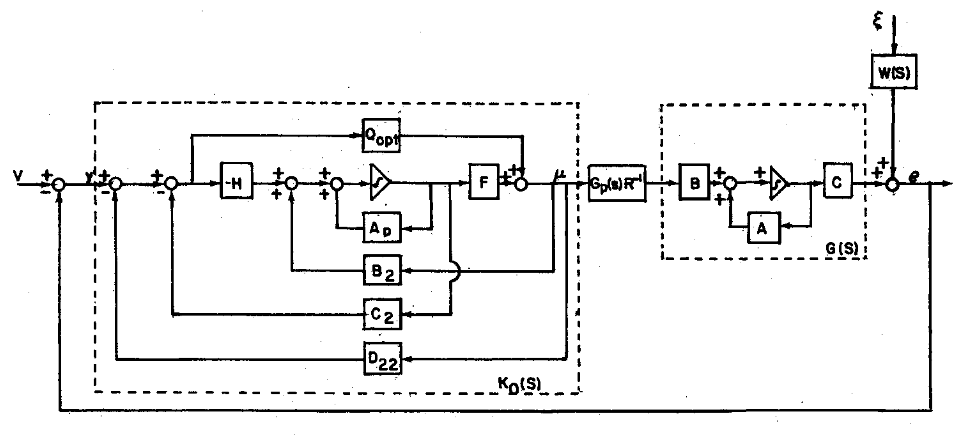

Fig. 6 Configuration of an H_2-optimal compensator.

Selecting a loop shape for a multivariable system is not a trivial task and it cannot be chosen in a single attempt. A control system designer, typically, tries several loop shapes before selecting a final desired loop shape. This is where the advantage of the Hardy-space optimization lies. This technique requires only one iteration to provide a solution for each loop shape without involving arbitrarily large or small numbers. In contrast, the regular Loop-Transfer-Recovery procedure involves arbitrarily large or small numbers in the compensator and may require several iterations for each loop shape before the limit of the solution sequence becomes apparent.

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